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ON THE OSCILLATION PROPERTIES OF FIRST-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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ABSTRACT

For first order linear impulsive differential equations with a deviating argument a number of oscillation theorems are proved (of the type of the Sturmian Theorem). Various criteria for oscillation or non-oscillation of the solutions of these equations are found. The oscillatory properties of some concrete equations of the type considered are investigated.

1. Introduction

The study of the oscillatory properties of differential equations and inequalities with deviating argument in recent years still arouses growing interest. Monographs [3], [7], [10] and [12] are devoted to this object.

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In relation to the oscillatory properties of impulsive differential equations, however, such literature is absent although these equations are being investigated very actively (see monographs [2], [4] and [5]). We shall just note the paper [9] where the problem itself is formulated and some initial results are obtained. In fact, the oscillatory properties of impulsive differential equations (without deviating argument) are described, so far as we know, in the single paper [1].

In the present paper a number of criteria are obtained for oscillation or non-oscillation of the solutions of first-order impulsive differential equations with deviating argument. This is done by extending to impulsive differential equations the Sturm-like Comparison Theory elaborated in [6] and [7, Chapter 4]. The oscillatory properties of some concrete impulsive differential equations with one retarded argument are also investigated in detail.

2. Main results

2.1. The Sturm-like Comparison Theory. Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. Consider the following two linear differential operators:

$$L[x] = x'(t) + \sum_{i=1}^{m} a_i(t)x[r_i(t)], \quad t \in \mathbb{R},$$

$$\tilde{L}[y] = -y'(t) + \sum_{i=1}^{m} q_i'(t)b_i[q_i(t)]y[q_i(t)], \quad t \in \mathbb{R}.$$

In particular, for $b_i(t) \equiv a_i(t)$, $i = 1, \ldots, m$, the corresponding operator $\tilde{L}$ is denoted by

$$L^*[y] = -y'(t) + \sum_{i=1}^{m} q_i'(t)a_i[q_i(t)]y[q_i(t)], \quad t \in \mathbb{R}.$$

Suppose that the following conditions (A) hold:

A1. The functions a_i, b_i: $\mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$ are piecewise continuous in $\mathbb{R}$.

A2. The functions r_i: $\mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$ are monotone increasing and continuously differentiable in $\mathbb{R}$, and the functions q_i: $\mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$ are their inverse ones: $r_i(q_i(t)) \equiv t$, $t \in \mathbb{R}$, $i = 1, \ldots, m$.

Let $J = (\xi, \eta)$ be a finite interval. Following [8] we define the subsequent sets:

$$r_i(J) = \Big\{t \in \mathbb{R}\colon\ t = r_i(s), \quad s \in J\Big\}, \quad i = 1, \dots, m,$$
$$q_i(J) = \Big\{t \in \mathbb{R}\colon\ t = q_i(s), \quad s \in J\Big\}, \quad i = 1, \dots, m,$$
$$F_{ext}(J) = \bigcup_{i=1}^{m} [r_i(J) \setminus J] = \Big(\bigcup_{i=1}^{m} r_i(J)\Big) \setminus J,$$
$$E_{ext}(J) = \bigcup_{i=1}^{m} [q_i(J) \setminus J] = \Big(\bigcup_{i=1}^{m} q_i(J)\Big) \setminus J.$$

In particular, if $r_i(t) \le t$ for $t \in \mathbb{R}$ and $\xi < r_i(\eta)$, $r_i(\xi) < \eta$, then

$$r_i(J) \setminus J = (r_i(\xi), \xi], \qquad q_i(J) \setminus J = [\eta, q_i(\eta)).$$

Let the sequence $\{t_j\}_{j=1}^{\infty}$ be given for which

$$(1) \qquad t_1 < t_2 < t_3 < \cdots, \qquad \lim_{j\to\infty} t_j = +\infty.$$

Consider the linear impulsive differential equation with deviating argument

$$(2.\text{i}) \qquad L[x] = 0, \qquad t \ne t_j,$$
$$(2.\text{ii}) \qquad x(t_j^+) = \alpha_j x(t_j^-)$$

and the associated impulsive differential inequalities

$$(3.\text{i}) \qquad L[x] \le 0, \qquad t \ne t_j,$$
$$(3.\text{ii}) \qquad x(t_j^+) = \alpha_j x(t_j^-)$$

and

$$(4.\text{i}) \qquad \tilde{L}[y] \ge 0, \qquad t \ne t_j,$$
$$(4.\text{ii}) \qquad y(t_j^+) = \beta_j y(t_j^-).$$

Let $\xi < \eta \le \infty$ and $J = (\xi, \eta)$.

Definition 1: The function $x(t)$ is said to be a **solution** of equation (2) in the interval J if:

1. $x(t)$ is defined in $J \cup F_{ext}(J)$.

2. $x(t)$ is absolutely continuous in each interval $(t_i, t_{i+1}) \cap J$, $i = 1, 2, \dots$, and it satisfies (2.i) for $t \in J$, $t \ne t_j$ almost everywhere.

3. $x(t)$ satisfies (2.ii) for $t = t_j \in J$.

The solution of the inequality (3) is defined analogously.

Definition 2: The function $y(t)$ is said to be a **solution** of inequality (4) in the interval J if:

1. $y(t)$ is defined in $J \cup E_{ext}(J)$.
2. $y(t)$ is absolutely continuous in each interval $(t_i, t_{i+1}) \cap J$, $i = 1, 2, \ldots$, and it satisfies (4.i) for $t \in J$, $t \neq t_j$ almost everywhere.
3. $y(t)$ satisfies (4.ii) for $t = t_j \in J$.

Henceforth, we suppose for the sake of definiteness that the solutions of equation (2) and inequalities (3) and (4) are continuous from the left at $t = t_j \in J$: $x(t_j^-) = x(t_j)$ $(y(t_j^-) = y(t_j))$.

Definition 3: (See [13]) The finite interval $J = (\xi, \eta)$ is said to be a **large positive hemicycle** of inequality (4) if

$$r_i(\eta) > \xi, \quad r_i(\xi) < \eta, \qquad i = 1, \ldots, m, \tag{5}$$

and there exists a solution $y(t)$ of inequality (4) in the interval J such that

$$y(\xi^+) = y(\eta^-) = 0; \quad y(t) > 0, \; t \in J.$$

The notions of **large positive hemicycle** for equation (2) and inequality (3) are defined analogously.

Definition 4: (See [7]) The finite interval $J = (\xi, \eta)$ is said to be a **regular positive hemicycle** of inequality (4) if relations (5) are valid and there exists a solution $y(t)$ of inequality (4) such that

$$y(\xi^+) = y(\eta^-) = 0; \quad y(t) > 0, \; t \in J; \quad y(t) \leq 0, \; t \in E_{ext}(J). \tag{6}$$

In this case, the interval $J \cup E_{ext}(J)$ is said to be an **extended regular hemicycle** of inequality (4).

Definition 5: The finite interval $J = (\xi, \eta)$ is said to be a **regular positive hemicycle** of equation (2) (or inequality (3)) if relations (5) are valid and there exists a solution $x(t)$ of equation (2) (or inequality (3)) in the interval J such that:

$$x(\xi^+) = x(\eta^-) = 0; \quad x(t) > 0, \; t \in J; \quad x(t) \leq 0, \; t \in F_{ext}(J).$$

In this case, the interval $J \cup F_{ext}(J)$ is said to be an **extended regular hemicycle** of equation (2) (or inequality (3)).

Example 1: The interval $J = (0,\pi)$ is a regular positive hemicycle for the equation

$$x'(t) + x\left(t - \frac{\pi}{2}\right) = 0$$

since $x = \sin t$ is a solution of this equation and

$$r_1(t) = t - \frac{\pi}{2}, \quad r_1(\pi) = \frac{\pi}{2} > 0, \quad \pi > r_1(0) = -\frac{\pi}{2},$$

$$x(0) = x(\pi) = 0; \quad x(t) > 0,\ t \in J; \quad x(t) \leq 0,\ t \in F_{ext}(J) = (-\frac{\pi}{2}, 0].$$

The results obtained below are based on the following

LEMMA 1: *Let $J = (\xi,\eta)$ be a finite interval and:*

1. *The function $x(t)$ is piecewise continuous in $J \cup F_{ext}(J)$.*
2. *The function $y(t)$ is piecewise continuous in $J \cup E_{ext}(J)$.*
3. *$x(t)$ and $y(t)$ are absolutely continuous in each interval $(t_i, t_{i+1}) \cap J$, $i = 1, 2, \ldots$, and for $t = t_j \in J$ they satisfy the relations*

$$x(t_j^+) = \alpha_j x(t_j^-), \quad y(t_j^+) = \beta_j y(t_j^-). \tag{7}$$

Then

$$\int_J \{y(t)L[x(t)] - x(t)\tilde{L}[y(t)]\}dt = \tag{8}$$

$$\sum_{i=1}^{m}\Big\{ \int_{J\cap q_i(J)} [a_i(t) - b_i(t)]x[r_i(t)]y(t)dt + \int_{J \setminus q_i(J)} a_i(t)x[r_i(t)]y(t)dt$$

$$- \int_{q_i(J)\setminus J} b_i(t)x[r_i(t)]y(t)dt\Big\}$$

$$+ \sum_{\xi<t_j<\eta} x(t_j^-)y(t_j^-)(1-\alpha_j\beta_j) + x(\eta^-)y(\eta^-) - x(\xi^+)y(\xi^+).$$

Proof: We obtain formula (8) taking into account the relations

$$\int_J \{yL[x] - x\tilde{L}[y]\}dt = \int_J (x'y + xy')dt \tag{9}$$

$$+\sum_{i=1}^{m}\int_J a_i x[r_i]ydt - \sum_{i=1}^{m}\int_J q_i' b_i[q_i]xy[q_i]dt,$$

$$x(\eta^-)y(\eta^-) - x(\xi^+)y(\xi^+) = \int_J (x'y + xy')dt$$

$$+ \sum_{\xi<t_j<\eta} \left[x(t_j^+)y(t_j^+) - x(t_j^-)y(t_j^-)\right],$$

relation (7) and changing the variables $q_i(t) = s$ in the integrals of the last sum in (9). ∎

THEOREM 1 (Sturm-like Comparison Theorem): *Suppose that:*

1. *The interval* $J = (\xi, \eta)$ *is a regular positive hemicycle for the inequality* (4).
2. *The following inequalities are valid:*

$$(10) \qquad b_i(t) \geq 0, \qquad t \in q_i(J) \setminus J, \quad i = 1, \dots, m,$$

$$(11) \qquad a_i(t) \geq 0, \qquad t \in J \setminus q_i(J), \quad i = 1, \dots, m,$$

$$(12) \qquad a_i(t) \geq b_i(t), \qquad t \in J \cap q_i(J), \quad i = 1, \dots, m,$$

$$(13) \qquad 1 - \alpha_j \beta_j \geq 0, \qquad t_j \in J.$$

3. *At least one of the inequalities* (13) *is strict or at least one of the inequalities* (10) – (12) *is strict in some subinterval of the respective sets* $q_i(J) \setminus J$, $J \setminus q_i(J)$, $J \cap q_i(J)$.

Then:

1. *Inequality* (3) *has no solution* $x(t)$ *which is positive in* $J \cup F_{ext}(J)$.
2. *Equation* (2) *has no solution* $x(t)$ *which preserves its sign in* $J_1 = J \cup F_{ext}(J)$.

Proof: 1. Suppose that Assertion 1 of Theorem 1 is not true, i.e., there exists a solution $x(t)$ of inequality (3) which is positive in $J \cup F_{ext}(J)$.

Since J is a regular positive hemicycle for inequality (4), then there exists a solution $y(t)$ of (4) which satisfies (6).

The solutions $x(t)$ and $y(t)$ satisfy the conditions of Lemma 1, thus they satisfy relation (8).

The left-hand side of (8) is nonpositive by (3), (4) and (6).

In view of conditions 2 and 3 of Theorem 1 and the relations

$$t \in J \cap q_i(J) \Longrightarrow r_i(t) \in J \Longrightarrow y(t) > 0, \quad x[r_i(t)] > 0;$$

$$t \in J \setminus q_i(J) \Longrightarrow r_i(t) \in F_{ext}(J) \Longrightarrow y(t) > 0, \quad x[r_i(t)] > 0;$$

$$t \in q_i(J) \setminus J \Longrightarrow t \in E_{ext}(J), \ r_i(t) \in J \Longrightarrow y(t) \leq 0, \quad x[r_i(t)] > 0$$

we conclude that the right-hand side in (8) is positive, which leads to a contradiction. Thus Assertion 1 is proved.

2. If we suppose that equation (2) has a solution $\varphi(t)$ which is negative in $J \cup F_{ext}(J)$, then $x = -\varphi(t)$ will be a solution of equation (2) which is positive

in $J \cup F_{ext}(J)$, but this is impossible by Assertion 1. Consequently, equation (2) has no solution which preserves its sign in $J \cup F_{ext}(J)$. ∎

Remark 1: Assertion 2 of Theorem 1 means that each solution $x(t)$ of equation (2) which is defined in $J_1 = J \cup F_{ext}(J)$ either changes its sign in J_1 or has at least one zero in J_1.

In the case when $\alpha_j > 0$, $t_j \in J_1$, then each such solution $x(t)$ has at least one zero in J_1 because then it is not possible for $x(t)$ to change its sign in $t_j \in J_1$ if $x(t_j) \neq 0$.

In the case when $\alpha_j < 0$ for some $t_j \in J_1$, the solution $x(t)$ of equation (2) may change its sign at $t = t_j$ without vanishing anywhere in J_1.

Example 2: Consider the equations

$$\begin{aligned} x'(t) + x\left(t - \frac{\pi}{2}\right) &= 0, \qquad t \in \mathbb{R},\ t \neq \frac{\pi}{2}j, \\ x\left(\frac{\pi}{2}j^+\right) &= -e^{-\pi/2}x(\frac{\pi}{2}j), \qquad j \in \mathbb{Z} \end{aligned} \tag{14}$$

and

$$\begin{aligned} -y'(t) + y\left(t + \frac{\pi}{2}\right) &= 0, \qquad t \in \mathbb{R},\ t \neq \frac{\pi}{2}j, \\ y\left(\frac{\pi}{2}j^+\right) &= y\left(\frac{\pi}{2}j\right), \qquad j \in \mathbb{Z}. \end{aligned} \tag{15}$$

Here $a_1(t) \equiv b_1(t) \equiv 1$, $\beta_j \equiv 1$, $\alpha_j = -e^{-\pi/2} < 0$, $r_1(t) = t - \pi/2$, $q_1(t) = t + \pi/2$.

It is easy to check that equation (15) has a solution $y = \cos t$ and $J = (-\frac{\pi}{2}, \frac{\pi}{2})$ is a regular positive hemicycle, and $F_{ext}(J) = (-\pi, -\frac{\pi}{2}]$, $E_{ext}(J) = [\frac{\pi}{2}, \pi)$.

By Theorem 1, equation (14) has no solution $x(t)$ which preserves its sign in $J_1 = J \cup F_{ext}(J) = (-\pi, \frac{\pi}{2})$.

A verification shows that the function

$$x(t) = (-1)^j e^{t - \frac{\pi}{2}j}, \quad t \in \left(\frac{\pi}{2}j, \frac{\pi}{2}(j+1)\right]$$

is a solution of equation (14) which is positive in each interval of the form $(\pi n, \pi n + \frac{\pi}{2}]$, $n \in \mathbb{Z}$, and is negative in the intervals $(\pi k - \frac{\pi}{2}, \pi k]$, $k \in \mathbb{Z}$, i.e., $x(t)$ is a solution which changes its sign without vanishing anywhere.

Consider the equation

$$\begin{aligned} \tilde{L}[y] &= 0, \quad t \neq t_j, \\ y(t_j^+) &= \beta_j y(t_j^-) \end{aligned} \tag{16}$$

and the inequalities associated with it

$$\tilde{L}[y] \le 0, \quad t \ne t_j, \qquad (17)$$
$$y(t_j^+) = \beta_j y(t_j^-)$$

and

$$L[x] \ge 0, \quad t \ne t_j, \qquad (18)$$
$$x(t_j^+) = \alpha_j x(t_j^-).$$

THEOREM 2: *Suppose that:*

1. *The interval $J = (\xi, \eta)$ is a regular positive hemicycle for inequality* (18).
2. *The following inequalities are valid:*

$$a_i(t) \ge 0, \qquad t \in J \setminus q_i(J), \quad i = 1, \ldots, m, \qquad (19)$$
$$b_i(t) \ge 0, \qquad t \in q_i(J) \setminus J, \quad i = 1, \ldots, m, \qquad (20)$$
$$b_i(t) \ge a_i(t), \qquad t \in J \cap q_i(J), \quad i = 1, \ldots, m, \qquad (21)$$
$$1 - \alpha_j \beta_j \le 0, \qquad t_j \in J. \qquad (22)$$

3. *At least one of the inequalities* (22) *is strict or at least one of the inequalities* (19) – (21) *is strict in some subinterval of the respective sets $J \setminus q_i(J)$, $q_i(J) \setminus J$, $J \cap q_i(J)$.*

Then:

1. *Inequality* (17) *has no solution $y(t)$ which is positive in $J \cup E_{ext}(J)$.*
2. *Equation* (16) *has no solution $y(t)$ which preserves its sign in $J \cup E_{ext}(J)$.*

The proof of Theorem 2 is analogous to the proof of Theorem 1.

As is known, the Sturmian Zeroes-Separation Theorem in its classical formulation is not valid for differential equations with a *deviating argument.* An exception is just a narrow class of *delay differential equations* of second order whose manifolds of solutions are *two-dimensional* (see [6]). This class does not contain quite usual delay differential equations with constant coefficients and constant delays. In [7] and [8] an essentially new approach is suggested for the formulation of the respective analogue of this theorem for delay differential equations.

Below, this approach is extended to impulsive delay differential equations.

Obviously conditions (10)–(13) of Theorem 1 and (19)–(22) of Theorem 2 are met if the following conditions are satisfied:

$$a_i(t) \equiv b_i(t) > 0, \quad t \in \mathbb{R}, \ i = 1, \ldots, m; \qquad 1 - \alpha_j \beta_j = 0, \ j \in \mathbb{N}. \qquad (23)$$

As a corollary of Theorems 1 and 2 we obtain the following assertion:

THEOREM 3 (The Sturm-like Zeroes-Separation Theorem): *Let conditions* (23) *hold.*

Then the extended regular hemicycle of one of the equations

$$L[x]=0, \quad t\neq t_j, \qquad x(t_j^+)=\alpha_j x(t_j^-), \tag{24}$$

$$L^*[y]=0, \quad t\neq t_j, \qquad y(t_j^+)=\alpha_j^{-1} y(t_j^-) \tag{25}$$

cannot be contained in a large positive hemicycle of the other equation.

Proof: The converse assumption leads to a contradiction with one of Theorems 1 and 2. ∎

Based on Theorems 1, 2 and 3, we shall formulate some analogues of Sturmian Oscillation and Non-oscillation Theorems for the scalar linear differential equations with deviating argument.

Let $x(t)$ be a solution of equation (2) defined for $t\geq\xi$.

Definition 6: The solution $x(t)$ of equation (2) is said to be **nonoscillating** if there exists $T\geq\xi$ such that $x(t)$ preserves its sign for $t>T$.

Definition 7: The solution $x(t)$ of equation (2) is said to be **oscillating** if for any $T\geq\xi$, $x(t)$ does not preserve its sign for $t>T$.

Definition 8: The solution $x(t)$ of equation (2) is said to be **regularly oscillating** if for any $T\geq\xi$ there exists a regular positive hemicycle J for equation (2) such that $J\subset(T,\infty)$.

The notions of nonoscillating, oscillating and regularly oscillating solutions of equations (16), (24) and (25) are defined analogously.

As a corollary of Theorem 1 we obtain

THEOREM 4 (The Sturm-like Oscillation Theorem): *Let the intervals* $J_n=(\xi_n,\eta_n)$ *with* $\lim_{n\to\infty}\xi_n=+\infty$ *be regular positive hemicycles of equation* (16) *and on each of them let conditions* 2, 3 *of Theorem* 1 *hold.*

Then all solutions of equation (2) *are oscillating.*

We shall note that the assumption of Theorem 4 is fulfilled if equation (16) has a regular oscillating solution. In view of this fact we deduce the following corollaries of Theorems 1, 2 and 4.

COROLLARY 1: *For $t \geq T$ let conditions* 2 *and* 3 *of Theorem* 1 *hold. Then:*

1. *If equation* (16) *has a regular oscillating solution, then all solutions of equation* (2) *are oscillating.*
2. *If equation* (2) *has a nonoscillating solution, then equation* (16) *has no regular oscillating solution.*

COROLLARY 2: *For $t \geq T$ let conditions* 2 *and* 3 *of Theorem* 2 *hold. Then:*

1. *If equation* (16) *has a nonoscillating solution, then equation* (2) *has no regular oscillating solution.*
2. *If equation* (2) *has a regular oscillating solution, then all solutions of equation* (16) *are oscillating.*

These corollaries applied to equations (24) and (25) take on the form:

COROLLARY 3: *Let conditions* (23) *hold. Then:*

1. *If equation* (25) *has a nonoscillating solution, then equation* (24) *has no regular oscillating solution.*
2. *If equation* (24) *has a nonoscillating solution, then equation* (25) *has no regular oscillating solution.*
3. *If equation* (25) *has a regular oscillating solution, then all solutions of equation* (24) *are oscillating.*
4. *If equation* (24) *has a regular oscillating solution, then all solutions of equation* (25) *are oscillating.*

2.2. FORMATION OF A BANK OF "STANDARD" EQUATIONS. The theory expounded above can be applied to obtaining effective criteria for oscillation of all solutions of equations of the form (2) and to the estimation of the length of the intervals in which these solutions preserve their sign only if sufficiently many "standard" equations of type (16) having a regular oscillating solution have been studied. Moreover, the criterion for existence of such a solution must be easy to verify. In this subsection we shall investigate the oscillatory properties of equations of types (2) and (16) having *one retarded argument.*

Consider the equation of the type (2)

$$x'(t) + a(t)x[r(t)] = 0, \quad t \neq t_j, \qquad x(t_j^+) = \alpha_j x(t_j^-) \tag{26}$$

having one retarded argument $r(t)$:

$$r(t) \leq t, \quad t \in \mathbb{R}, \qquad \lim_{t\to\infty} r(t) = +\infty.$$

As above we suppose that the moments $\{t_j\}_{j=1}^{\infty}$ satisfy (1) and that the function $r(t)$ is a continuously differentiable and monotone increasing function having an inverse one $q(t)$: $r[q(t)] \equiv t$, $t \in \mathbb{R}$.

The role of equation (16) is played by the equation

$$\begin{aligned} -y'(t) + q'(t)b[q(t)]y[q(t)] = 0, \quad t \neq t_j, \\ y(t_j^+) = \beta_j y(t_j^-). \end{aligned} \tag{27}$$

The next lemma provides the possibility to construct equations of the type (27) having a regularly oscillating solution.

LEMMA 2: *Let the finite interval $J = (\xi, \eta)$ and the function $\varphi(t)$ satisfy the following conditions:*

1. *$\beta_j > 0$ for $t_j \in (\xi, \eta)$ and $r(\eta) > \xi$.*
2. *$\varphi(t)$ is continuous in $(r(\xi), q(\eta))$ and*

$$\int_{\xi}^{\eta} \varphi(s)ds = \pi, \tag{28}$$

$$0 < \int_{\xi}^{t} \varphi(s)ds < \pi, \quad t \in (\xi, \eta), \tag{29}$$

$$0 \leq \int_{\eta}^{t} \varphi(s)ds \leq \pi, \quad t \in [\eta, q(\eta)), \tag{30}$$

$$0 < \int_{r(t)}^{t} \varphi(s)ds < \pi, \quad t \in (\xi, q(\eta)). \tag{31}$$

Then the interval $J = (\xi, \eta)$ is a regular positive hemicycle for equation (27) in which

$$b(t) = \prod_{r(t) \leq t_j < t} \beta_j^{-1} \frac{r'(t)\varphi[r(t)]}{\sin \int_{r(t)}^{t} \varphi(s)ds} \exp\left\{ -\int_{r(t)}^{t} \varphi(s) \cot\left(\int_{s}^{q(s)} \varphi(z)dz \right) ds \right\}. \tag{32}$$

Proof: From condition 1 and (31) it follows that the function $b(t)$ is defined in $(\xi, q(\eta)) = J \cup E_{ext}(J)$.

A straightforward verification shows that the function

$$y(t) = \prod_{t_j<t} \beta_j \exp\left\{\int_\xi^t \varphi(s)\cot\left(\int_s^{q(s)} \varphi(z)dz\right)ds\right\}\sin\int_\xi^t \varphi(s)ds \tag{33}$$

is a solution of equation (27) in J.

Taking into account condition 1, (28) and (29), we conclude that J is a large positive hemicycle of equation (27).

Since by condition 1 and (30) it follows that $y(t) \le 0$, $t \in E_{ext}(J) = [\eta, q(\eta))$, then J is a regular positive hemicycle of equation (27). ∎

For construction of various equations (27) having a nonoscillating solution we shall use the following

Lemma 3: *Let the function $\psi(t)$ be continuous for $t \ge T$ and in equation (27) let $\beta_j > 0$, $t_j \ge T$ and let $b(t)$ have the form*

$$b(t) = \prod_{r(t)\le t_j<t} \beta_j^{-1} r'(t)\psi[r(t)]\exp\left(-\int_{r(t)}^t \psi(s)ds\right), \qquad t \ge T.$$

Then equation (27) has a nonoscillating solution

$$y(t) = \prod_{T\le t_j<t} \beta_j \exp\left(\int_T^t \psi(s)ds\right). \tag{34}$$

Lemma 3 is proved by a straightforward verification.

Theorem 5: *Suppose that:*

1. *The finite interval $J = (\xi, \eta)$ and the functions $r(t)$ and $\varphi(t)$ satisfy the conditions of Lemma 2 and*

$$\varphi(t) \ge 0, \quad t \in (r(\xi), \xi] \cup [r(\eta), \eta). \tag{35}$$

2. *In equation (26) the sequence $\{\alpha_j\}$ is positive and together with the function $a(t)$ satisfies the inequality*

(36)

$$a(t)\prod_{r(t)\le t_j<t} \alpha_j^{-1} \ge \frac{r'(t)\varphi[r(t)]}{\sin\int_{r(t)}^t \varphi(s)ds}\exp\left\{-\int_{r(t)}^t \varphi(s)\cot\left(\int_s^{q(s)} \varphi(z)dz\right)ds\right\}$$

for $t \in (\xi, \eta)$.

3. *One of the inequalities* (35) *or* (36) *is strict in some subinterval of the respective sets* $(r(\xi), \xi] \cup [r(\eta), \eta)$ *and* (ξ, η).

Then each solution of equation (26) *has at least one zero in the interval* $(r(\xi), \eta) = J \cup F_{ext}(J)$.

Proof: Theorem 5 is a corollary of Theorem 1 and Lemma 2 with $\beta_j = \alpha_j^{-1}$. ∎

An immediate corollary of Theorem 5 is the following assertion:

COROLLARY 4: *Let the function* $\varphi(t)$ *and equation* (26) *be defined for* $t \geq T$ *and the conditions of Theorem* 5 *hold for the sequence of finite intervals* $J_n = (\xi_n, \eta_n)$ *with* $\lim_{n\to\infty} \xi_n = +\infty$.

Then all solutions of equation (26) *are oscillating. Moreover, each solution of* (26) *has at least one zero in the interval* $(r(\xi_n), \eta_n)$.

Sometimes it is convenient to use the following particular case of Corollary 4.

COROLLARY 5: *Assume that:*

1. *The function* $\varphi(t)$ *is continuous for* $t \geq T$ *and*

$$\varphi(t) \geq 0, \quad t \geq T, \qquad \int^{\infty} \varphi(s)ds = \infty, \tag{37}$$

$$0 < \int_{r(t)}^{t} \varphi(s)ds < \pi, \quad t \geq T. \tag{38}$$

2. *In equation* (26) *the sequence* $\{\alpha_j\}$ *is positive for* $t_j \geq T$ *and together with the function* $a(t)$ *satisfies inequality* (36) *for* $t \geq T$.

Then all solutions of equation (26) *are oscillating.*

THEOREM 6: *Assume that:*

1. *The function* $\psi(t)$ *is nonnegative and continuous for* $t \geq T$.
2. *In equation* (26) *the sequence* $\{\alpha_j\}$ *is positive for* $t_j \geq T$ *and together with the function* $a(t)$ *satisfies the inequality*

$$0 \leq a(t) \prod_{r(t)\leq t_j<t} \alpha_j^{-1} \leq r'(t)\psi[r(t)]\exp\left(-\int_{r(t)}^{t} \psi(s)ds\right), \quad t \geq T. \tag{39}$$

3. *One of the two inequalities in* (39) *is strict.*

Then equation (26) *has no regular oscillating solution.*

Proof: Theorem 6 is a consequence of Corollary 2 and Lemma 3 with $\beta_j = \alpha_j^{-1}$ applied to equations (26) and (27). ∎

We shall note that for a different choice of the functions φ and ψ in Theorems 5 and 6 we can obtain different criteria for oscillation of all solutions or absence of a regular oscillating solution of equation (26). Let us now consider some of these.

Consider the impulsive equation with constant delay

$$\begin{aligned} x'(t) + a(t)x(t-\sigma) = 0, \quad t \geq 0, \quad t \neq t_j, \\ x(t_j^+) = \alpha_j x(t_j^-) \end{aligned} \tag{40}$$

where $\sigma > 0$ and $\alpha_j > 0$ for $t_j \geq 0$.

Set in Theorem 5 $\varphi(t) \equiv \nu$, $t \geq 0$. Then as a particular case of Theorem 5 we deduce

COROLLARY 6: *Let $\alpha_j > 0$ for $t_j \geq 0$. Then:*

1. *If* $0 < \nu < \frac{\pi}{\sigma}$ *and in some interval* $(\tau_0, \tau_0 + \frac{\pi}{\nu})$ *we have*

$$a(t) \prod_{t-\sigma \leq t_j < t} \alpha_j^{-1} \geq \frac{\nu}{\sin(\nu\sigma)} \exp\big(-\nu\sigma\cot(\nu\sigma)\big), \tag{41}$$

then each solution of equation (40) *has at least one zero in* $(\tau_0 - \sigma, \tau_0 + \frac{\pi}{\nu})$.

2. *If there exists a sequence* $\{\nu_n\}$, $0 < \nu_n < \frac{\pi}{\sigma}$ *and intervals* $(\tau_n, \tau_n + \frac{\pi}{\nu_n}) \subset \mathbb{R}_+$ *with* $\lim_{n\to\infty} \tau_n = +\infty$, *in which* (41) *is met, with* $\nu = \nu_n$, *then each solution of equation* (40) *oscillates and has at least one zero in each of the intervals* $(\tau_n - \sigma, \tau_n + \frac{\pi}{\nu_n})$.

COROLLARY 7: *Let $\alpha_j > 0$ for $t_j \geq 0$ and let the following condition hold:*

$$\liminf_{t\to+\infty} \left\{ a(t) \prod_{t-\sigma \leq t_j < t} \alpha_j^{-1} \right\} > \frac{1}{e\sigma}. \tag{42}$$

Then all solutions of equation (40) *are oscillating.*

Proof: Corollary 7 follows from Corollary 6 and the fact that

$$\lim_{\mu\to 0} \frac{\mu}{\sin\mu} \exp\big(-\mu\cot\mu\big) = \frac{1}{e}. \qquad ∎$$

Set $\psi(t) = 1/\sigma$ in Theorem 6 and obtain

COROLLARY 8: *Let $\alpha_j > 0$ for $t_j \geq T$ and for $t \geq T$ let the condition*

$$0 \leq a(t) \prod_{t-\sigma \leq t_j < t} \alpha_j^{-1} \leq \frac{1}{e\sigma} \tag{43}$$

hold and let one of the inequalities in (43) be strict.

Then equation (40) has no regular oscillating solution.

Consider the equation

$$\begin{aligned} x'(t) + a(t)x\left(\frac{t}{\lambda}\right) &= 0, \quad t \neq t_j, \\ x(t_j^+) &= \alpha_j x(t_j^-) \end{aligned} \tag{44}$$

for $t > 0$, where $\lambda > 1$ and $\alpha_j > 0$ for $t_j > 0$.

Equation (44) is a typical representative of an equation with unbounded increasing delay ($\lim_{t\to+\infty}[t - r(t)] = +\infty$). The oscillatory properties of this equation in the case without impulses ($\alpha_j \equiv 1$) have been studied before, for instance in [7], [11], [14].

Set in Theorem 5 $\varphi(t) = \nu/t$, $0 < \nu < \pi/\log\lambda$. Then we have

$$r(t) = \frac{t}{\lambda}, \quad q(t) = \lambda t, \qquad \int_{r(t)}^{t} \varphi(s)ds = \int_{t}^{q(t)} \varphi(z)dz = \nu\log\lambda$$

and

$$\int_{\xi}^{\eta} \varphi(s)ds = \pi \quad \Longleftrightarrow \quad \eta = \xi\exp\frac{\pi}{\nu}.$$

Inequality (36) takes the form

$$ta(t) \prod_{t/\lambda \leq t_j < t} \alpha_j^{-1} \geq \frac{\nu}{\sin(\nu\log\lambda)}\exp\big(- \nu\log\lambda . \cot(\nu\log\lambda)\big). \tag{45}$$

Consequently, the following assertion is valid:

COROLLARY 9: *Let $\alpha_j > 0$ for $t_j > 0$. Then:*

1. *If $0 < \nu < \pi/\log\lambda$ and inequality (45) is valid in some interval $(\tau_0, \tau_0\exp\frac{\pi}{\nu})$, then each solution of equation (44) has at least one zero in $(\frac{\tau_0}{\lambda}, \tau_0\exp\frac{\pi}{\nu})$.*
2. *If there exists a sequence $\{\nu_n\}$, $0 < \nu_n < \pi/\log\lambda$ and intervals $(\tau_n, \tau_n\exp\frac{\pi}{\nu_n}) \subset \mathbb{R}_+$ with $\lim_{n\to\infty}\tau_n = +\infty$, in which (45) is valid, with $\nu = \nu_n$, then each solution of equation (44) oscillates and has at least one zero in each of the intervals $(\frac{\tau_n}{\lambda}, \tau_n\exp\frac{\pi}{\nu_n})$.*

In particular, passing to the limit in (45), we obtain

COROLLARY 10: *Let* $\alpha_j > 0$ *for* $t_j > 0$ *and*

(46)
$$\liminf_{t\to+\infty}\left\{ta(t)\prod_{t/\lambda\le t_j<t}\alpha_j^{-1}\right\} > \frac{1}{e\log\lambda}.$$

Then all solutions of equation (44) *oscillate.*

The constant $1/(e\log\lambda)$ in (46) is the best as shown by the following Corollary 11 — a particular case of Theorem 6 with $\psi(t) = 1/(t\log\lambda)$.

COROLLARY 11: *Let* $\alpha_j > 0$ *for* $t_j \ge T > 0$ *and*

(47)
$$0 \le ta(t)\prod_{t/\lambda\le t_j<t}\alpha_j^{-1} \le \frac{1}{e\log\lambda}, \quad t \ge T$$

and let one of the inequalities in (47) *be strict.*

Then equation (44) *has no regularly oscillating solution.*

Remark 2: We shall note that in Corollaries 7–10 the positive sequence $\{\alpha_j\}$ and the function $a(t)$ jointly influence the oscillatory properties of the respective equations, and their influences can compensate for each other. That is why $\{\alpha_j\}$ can both be unbounded and satisfy $\liminf_{j\to\infty}\alpha_j = 0$.

Also in these corollaries there are no restrictions of the type

(48)
$$\alpha_j \ge 1, \qquad \sum^{\infty}(\alpha_j - 1) < \infty$$

which are imposed in [9]. In our opinion, condition (48) arises not from the problem itself but from the method for solving it used in [9].

Consider again equation (26). Let $a_0(t)$ be a continuous function and

(49)
$$0 \le a_0(t) \le a(t), \quad t \ge T.$$

Set in (36)

$$\varphi(t) = \nu q'(t)a_0[q(t)], \quad 0 < \nu < \nu_0$$

and by a change of variables this condition takes the form

(50)
$$a(t)\prod_{r(t)\le t_j<t}\alpha_j^{-1} \ge \frac{\nu a_0(t)}{\sin\left(\nu\int_t^{q(t)}a_0(s)ds\right)}\exp\left(-\int_t^{q(t)}\frac{\nu a_0(s)ds}{\tan\left(\nu\int_s^{q(s)}a_0(z)dz\right)}\right)$$

Taking into account (49), we conclude that (50) is met if

(51)
$$F(\nu,t)\equiv\frac{1}{\nu}\prod_{r(t)\le t_j<t}\alpha_j^{-1}\sin\left(\nu\int_t^{q(t)}a_0(s)ds\right)\exp\left(\int_t^{q(t)}\frac{\nu a_0(s)ds}{\tan\left(\nu\int_s^{q(s)}a_0(z)dz\right)}\right)\ge 1.$$

The interval $J=(\xi,\eta)$ which satisfies condition (28) must be such that

$$\int_\xi^\eta\varphi(s)ds=\nu\int_\xi^\eta q'(s)a_0[q(s)]ds=\nu\int_{q(\xi)}^{q(\eta)}a_0(\xi)d\xi=\pi.$$

Thus we have proved the following theorem:

THEOREM 7: *Suppose that:*

1. *The sequence $\{\alpha_j\}$ is positive for $t_j\ge T$.*
2. *The function $a_0(t)$ is continuous for $t\ge T$ and*

$$0\le a_0(t)\le a(t),\qquad t\ge T,$$
$$0\le\int_t^{q(t)}a_0(s)ds\le M,\qquad t\ge T,$$
$$\int^\infty a_0(s)ds=\infty.$$

3. *In the interval $J=(\xi,\eta)$ condition (51) is valid with $\nu\in(0,\pi/M)$ and*

$$\int_{q(\xi)}^{q(\eta)}a_0(s)ds\ge\frac{\pi}{\nu}.$$

Then each solution of equation (26) has at least one zero in $(r(\xi),\eta)$.

COROLLARY 12: *Let conditions 1 and 2 of Theorem 7 hold and*

(52)
$$\liminf_{t\to+\infty}F(0+,t)\equiv$$
$$\equiv\liminf_{t\to+\infty}\left\{\prod_{r(t)\le t_j<t}\alpha_j^{-1}\int_t^{q(t)}a_0(s)ds\cdot\exp\left(\int_t^{q(t)}\frac{a_0(s)ds}{\int_s^{q(s)}a_0(z)dz}\right)\right\}>1.$$

Then all solutions of equation (26) *are oscillating.*

Proof: From (52) it follows that there exist $\delta > 0$ and $t_\delta > T$ such that $F(0+,t) \geq 1+\delta$ for $t \geq t_\delta$. On the other hand, $F(0+,t) = \lim_{\nu\to 0+} F(\nu,t)$. Thus there exists $\nu_0 > 0$ such that (51) is valid for $0 < \nu < \nu_0$. Then the assertion of Corollary 12 follows from Theorem 7. ∎

As a corollary of Theorem 6 we deduce the following assertion which shows that condition (52) is precise enough.

COROLLARY 13: *Suppose that:*

1. *The sequence* $\{\alpha_j\}$ *is positive for* $t_j \geq T$.
2. *The function* $a^0(t)$ *is continuous for* $t \geq T$ *and*

$$0 \leq a(t) \leq a^0(t), \quad t \geq T, \tag{53}$$
$$\int_t^{q(t)} a^0(s)ds > 0, \quad t \geq T,$$

$$\prod_{r(t)\leq t_j<t} \alpha_j^{-1} \int_{r(t)}^{t} a^0(s)ds \cdot \exp\left(\int_t^{q(t)} \frac{a^0(s)ds}{\int_{r(s)}^{s} a^0(z)dz}\right) \leq 1, \quad t \geq T. \tag{54}$$

3. *One of the inequalities in* (53) *or* (54) *is strict.*

Then equation (26) *has no regularly oscillating solution.*

Proof: In Theorem 6 set

$$\psi(t) = q'(t)a^0[q(t)]\left(\int_t^{q(t)} a^0(z)dz\right)^{-1}.$$

Then inequality (39) takes on form

$$a(t)\prod_{r(t)\leq t_j<t} \alpha_j^{-1} \leq a^0(t)\left(\int_{r(t)}^{t} a^0(s)ds\right)^{-1} \exp\left(-\int_t^{q(t)} \frac{a^0(s)ds}{\int_{r(s)}^{s} a^0(z)dz}\right)$$

and this inequality is valid for $t \geq T$ by virtue of (54). Consequently, by Theorem 6, equation (26) has no regularly oscillating solution. ∎

2.3. Oscillatory Properties of Nonlinear Impulsive Differential Equations with a Deviating Argument. In this subsection we shall apply some of the results obtained above to the study of the oscillatory properties of the nonlinear differential equation with one deviating argument

$$\begin{aligned} z'(t) + f(t, z(t), z[r(t)]) = 0, \quad t \ge T, \quad t \ne t_j, \\ z(t_j^+) = f_j(z(t_j^-)). \end{aligned} \tag{55}$$

Theorem 8: *Suppose that:*

1. *The function $r(t)$ is monotone increasing and continuously differentiable in $\mathbb{R}$, having an inverse one $q(t) : r[q(t)] \equiv t$, $t \in \mathbb{R}$.*
2. *The function $f(t,u,v)$ is such that if $z(t)$ is an absolutely continuous function in each interval $(t_i, t_{i+1}) \cap [T, \infty)$, $i = 1, 2, \dots$, then the function $f(t, z(t), z[r(t)])$ is piecewise continuous for $t \ge T$.*
3. *The interval $J = (\xi, \eta) \subset (T, \infty)$ is a regular positive hemicycle of equation (27).*
4. *The following inequalities are valid*

$$\begin{aligned} vf(t,u,v) \ge b(t)v^2, \quad t \in J, \quad (u,v) \in \mathbb{R}^2, \\ uf_j(u)\beta_j \le u^2, \quad t_j \in J, \quad u \in \mathbb{R}, \\ b(t) \ge 0, \quad t \in J \cup E_{ext}(J) \end{aligned} \tag{56}$$

and one of these inequalities is strict.

Then equation (55) has no solution which preserves its sign in $J \cup F_{ext}(J)$.

Proof: Suppose that this is not true, i.e., that equation (55) has a solution $z_0(t)$ which preserves its sign in $J \cup F_{ext}(J)$.

Consider the linear equation

$$\begin{aligned} x'(t) + \frac{f(t, z_0(t), z_0[r(t)])}{z_0[r(t)]} x[r(t)] = 0, \quad t \in J, \quad t \ne t_j, \\ x(t_j^+) = \frac{f_j(z_0(t_j^-))}{z_0(t_j^-)} x(t_j^-), \quad t_j \in J. \end{aligned} \tag{57}$$

Obviously, $x = z_0(t)$ is a solution of (57) in J. On the other hand,

$$a(t) = \frac{f(t, z_0(t), z_0[r(t)])}{z_0[r(t)]} \ge b(t), \quad t \in J,$$

$$\alpha_j\beta_j = \frac{f_j(z_0(t_j^-))}{z_0(t_j^-)}\beta_j \le 1, \quad t_j \in J.$$

Consequently, by Theorem 1, equation (57) has no solution which preserves its sign in $J \cup F_{ext}(J)$ and we are led to a contradiction. ∎

COROLLARY 14: *Suppose that:*

1. *Conditions* 1 *and* 2 *of Theorem* 8 *hold.*
2. *The inequalities* (56) *are valid for* $t_j \geq T$, $t \geq T$, $(u, v) \in \mathbb{R}^2$, *and one of these inequalities is strict.*
3. *Equation* (27) *has a regularly oscillating solution.*

Then all solutions of equation (55) *which are defined in an infinite half-interval* $[\xi, \infty) \subset [T, \infty)$ *are oscillating.*

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